THOM-PONTRJAGIN THEOREM

1. Differential topogology preliminaries

We review basic facts from differential topology needed for later use.

1.1. **Submanifolds.** In this section we carefully define submanifolds of manifolds with boundary. We restrict our attention to neat submanifolds and state how can they arise as inverse images of submersions. All manifolds considered are smooth manifolds with boundary unless otherwise stated.

Definition 1.1. Let M_1, M_2 and N_1, N_2 be manifolds, $x_i \in X_i$ and $y_i \in N_i$ and let $f_i : M_i \to N_i$ be smooth maps. We say $(M_1, N_1, x_1, y_1, f_1)$ *locally looks like* $(M_2, N_2, x_2, y_2, f_2)$ if there exist neighborhoods U_i of x_i and V_i of x_i and diffeomorphisms $g : U_1 \to U_2$ and $h : V_1 \to V_2$ such that

(1) $g(x_1) = x_2$, $h(y_1) = y_2$ and $f_i(U_i) \subset V_i$
(2) The following diagram is commutative

$$\begin{array}{ccc} U_1 & \xrightarrow{f_1} & V_1 \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ U_2 & \xrightarrow{f_2} & V_2 \end{array}$$

Lemma 1.2. *Let M and N be manifolds, $x \in M$ and $y = f(x)$. Suppose $f : M \to N$ is a smooth map.*

- *(Inverse function theorem) If df_x is isomorphism, and $f(\partial M) \subset \partial N$, then f is local diffeomorphism at x.*
- *(Submersion theorem) If f and ∂f are submersions at $x \in \partial X$ and $\partial Y = \emptyset$, then (M, N, x, y, f) looks locally like $(\mathbb{R}^m_+, \mathbb{R}^n, 0, 0, \pi)$ where*

$$\pi(x_1, \cdots, x_n, x_{n+1}, \cdots, x_m) = (x_1, \cdots, x_n)$$

Definition 1.3. Let S and M be manifolds such that $S \subset M$ is a manifold. We say S is *submanifold* of M if the natural inclusion $i_S : S \hookrightarrow M$ is a smooth embedding, so in particular the topology on S is the subspace topology.

The following theorem records some of the properties of submanifolds

Proposition 1.4 (Properties of submanifolds)**.** *Let X and Y be manifolds.*

(1) *A set map $f : N \to S$ is smooth if and only if $i_S \circ f$ is smooth.*
(2) *(Uniqueness) If there is a manifold structure on $S \subset X$ such that S is a submanifold then this manifold structure is unique. So when we say a set $S \subset X$ is submanifold of M, we mean S can be endowed with the unique manifold structure turning it into submanifold.*
(3) *(Open submanifolds) Every open subset $U \subset X$ is a submanifold of M.*
(4) *(Smooth maps) Any smooth map $f : S \to \mathbb{R}$ is the restriction of smooth map $f : M \to \mathbb{R}$.*

(5) *(Locality) $S \subset M$ is a submanifold of X if and only if for every $x \in S$, there is a neighborhood U around x such that $U \cap S$ is a submanifold of M.*
(6) *(Monotonicity) If S_1 is submanifold of S_2 and S_2 is submanifold of M, then S_1 is submanifold of X.*
(7) *(Local description) Let $S \subset M$ be a submanifold. For every $x \in S \cap \operatorname{int} X$, there is a chart (U, x, ϕ) such that $U \cap S = \phi^{-1}(\mathbb{R}^m)$.*

With this definition of submanifold, the behavior of the submanifold at the boundary is problematic. We need only the what-so-called "neat submanifolds", which behaves at boundary like the submanifold $\mathbb{R}^m_+$ of $\bar{\mathbb{R}}^n_+$. This motivates a definition

Definition 1.5. A submanifold $S \subset M$ is *neat submanifold* if the following condition is satisfied

(Neat) $S \cap \partial M = \partial S$ and for every $x \in S \cap \partial M$, there is a chart (U, ϕ) of M around x, where $\phi : U \to \mathbb{R}^n_+$ and $U \cap S = \phi^{-1}(0 \times \mathbb{R}^{m-n}_+)$.

As it will be seen neat submanifolds are the correct generalizations of submanifolds encountered in a first course of smooth manifolds. The following proposition gives us a condition to determine if the submanifold is neat.

Proposition 1.6. *A submanifold $S \subset M$ is neat if and only if $S \cap \partial M = \partial S$ and for any $x \in \partial S$, some inward pointing vector in $T_x(M)$ is an inward pointing vector of $T_x(M)$.*

Proposition 1.7. *Let M be a manifold. A subspace $S \subset M$ is a neat submanifold if and only if*

(I_1) *For every $x \in S \cap \operatorname{int} M$ there is a chart (U, x, ϕ) such that $S \cap U = \phi^{-1}(0 \times \mathbb{R}^{m-n})$*

(I_2) *For every $x \in S \cap \partial M$ there is a chart (U, x, ϕ) such that $S \cap U = \phi^{-1}(0 \times \mathbb{R}^{m-n}_+)$*

Proof. We need to check that for every $x \in S$ there is a neighborhood U of x such that $S \cap U$ is a submanifold. We do the case $x \in S \cap \partial M$. From condition (I_2), It is immediate that $\pi_2 \circ \phi$ is topological open embedding. Endow $S \cap U$ the smooth structure given by the chart $(U \cap S, x, \pi_2 \circ \phi)$, and note that the following diagram commutes

$$\begin{tikzcd}
U \cap S \arrow[r, hook] \arrow[d, "\pi_2 \circ \phi"] & U \arrow[d, "\phi"] \\
\mathbb{R}^{m-n}_+ \arrow[r, hook] & \mathbb{R}^m_+
\end{tikzcd}$$

Each map is smooth map. Clearly the two vertical maps are smooth embeddings, and the lower horizontal inclusion is also a smooth embedding, hence the inclusion $U \cap S \to U$ is smooth embedding. In other words $U \cap S$ is a submanifold of U. By properties of submanifolds $U \cap S$ is submanifold of S. $\square$

The version of regular level theorem is

Proposition 1.8 (Regular Value Theorem). *Let M and N be manifolds, where $\partial N = \emptyset$. Suppose $f : M \to N$ is smooth, and $y \in N$ is regular value of both f and ∂f. Then $f^{-1}(y)$ is a neat submanifold of M.*

Proof. We need to check (I_1) and (I_2). Let $x \in f^{-1}(y) \cap \partial X$, then f is a submersion at x. It follows by lemma 1.2 that (M, N, x, y, f) locally looks like $(\mathbb{R}^m_+, \mathbb{R}^n, 0, 0, \pi)$ where π is the projection. That's there are charts (U_x, x, ϕ_x) and (V_y, y, ψ_y) of M and N respectively, such that the following diagram commutes

$$\begin{array}{ccc} U_x & \xrightarrow{f} & V_y \\ \downarrow{\scriptstyle \phi_x} & & \downarrow{\scriptstyle \psi_y} \\ \mathbb{R}^m_+ & \xrightarrow{\pi} & \mathbb{R}^n \end{array}$$

As consequence $f^{-1}(y) \cap U_x = (\psi_y \circ f)^{-1}(0) = (\pi \circ \phi_x)^{-1}(0) = \phi_x^{-1}(0 \times \mathbb{R}^{m-n}_+)$. So the condition (I_1) is satisfied. The condition (I_2) is satisfied by a similar proof. □

An important construction is the following:

Definition 1.9. A *collar* on M is a pair (U, f) of an open set U containing ∂M, and diffeomorphism $f : \partial M \times [0, 1) \to U$ extending the natural inclusion $\partial M \to M$.

In fact each manifold with boundary admits a collar:

Theorem 1.10 (Existence of collars)**.** *Let S be a closed neat submanifold of M, then there is a collar of M which restricts to a collar of S. (i.e there is a collar (U, f) of M such that $(U \cap S, f|_{\partial S\times[0,1)})$ is a collar of S).*

Exercise. The proof is divided into three steps:

Step 1. Construct a vector field $v \in \Gamma(TM)$ satisfying:

- v restricts to a vector field on S, that's $v \circ i \in \Gamma(TS)$.
- v points inward along the boundary, that's $v|_{\partial M} \in T^+(\partial M)$.

Step 2. Construct a smooth map $\theta : D \to M$, such that

(1) $D = \{(x, t)|x \in \partial M\ ,\ 0 \le t < \delta(x)\}$ where $\delta : \partial M \to M$ is smooth.
(2) $\theta(x, -) : [0, \delta(x)) \to M$ is integral curve of v, with $\theta(x, 0) = x$.

Step 3. Construct the collar. Use the following fact from topology :

> *(*) Let $f : X \to Y$ be a local homomorphism between paracompact spaces. Suppose $C \subset X$ is a closed subset such that f is a homomorphism from C onto $f(C)$. Then there is an open subset U containing C, such that $f : U \to f(U)$ is homomorphism.*

□

1.2. **Embedding Theorems.**

Theorem 1.11 (Whitney)**.** *Let M be a manifold with boundary, then M can be smoothly and properly embedded into $\mathbb{R}^N$ for large enough N.*

Exercise. Using collars above one reduce to case M has no boundary. Now the proof for the latter case can be found in your favorite introductory book on smooth manifolds. □

We can use collars to modify the above theorem to a "neat" embedding theorem.

Corollary 1.12 (Neat Embedding)**.** *Let M be a manifold with disconnected compact boundary, $\partial M = N_0 \cup N_1$. Then M can be neatly embedded into $\mathbb{R}^N \times [0, 1]$ (understood as manifold with boundary $\mathbb{R}^N \times \{0, 1\}$) for large enough N, such that $N_k \subset \mathbb{R}^N \times \{i\}$*

Proof. Pick up an embedding $i : M \to \mathbb{R}^N$ by Whitney embedding theorem. Let (U, f) be a collar of ∂M, by possibly shrinking U, we can assume U is a disjoint union $U_0 \cup U_1$,where $f_k = f|_{N_k \times [0,1)}$ is diffeomorphism onto U_i. It is easy to construct a C^∞ function $\rho : [0, 1) \to [0, 1)$ which satisfies

- $\rho(t) = \begin{cases} 0, & \text{if } t \leq 1/4 \\ t, & \text{if } t \geq 1/2 \end{cases}$
- ρ is strictly increasing $[1/2, 1)$

and a C^∞ function $\tau : [0, 1) \to [0, 1/2]$ which satisfies

- $\tau(t) = \begin{cases} t & \text{if } t \leq 1/4 \\ 1/2 & \text{if } t \geq 1/2 \end{cases}$
- τ is strictly increasing $[0, 1/2)$

Define $j : U \to \mathbb{R}^N \times [0, 1]$,

$$j(f(x, t)) = \begin{cases} (i(f(x, \rho(t)), \tau(t)) & \text{if } x \in U_0 \\ (i(f(x, \rho(t)), 1 - \tau(t)), & \text{if } x \in U_1 \end{cases}$$

Clearly j is embedding. Define

$$\tilde{i}(x) = \begin{cases} j(x) & \text{if } x \in U \\ i(x) & \text{if } x \notin U \end{cases}$$

$\tilde{i}$ is the required embedding. □

We have the following important approximation theorem:

Theorem 1.13. *Let M be a manifold and $A \subset M$ be a closed subset of M. Suppose that $f : M \to \mathbb{R}^N$ is a continuous map such that $f|_A$ is smooth and that $\delta : M \to \mathbb{R}_{>0}$ be continuous function given. Then there is $\tilde{f} : M \to \mathbb{R}^N$ such that $|f - \tilde{f}| < \delta$.*

Proof. Since f is smooth on A, then we can find a smooth map $f_0 : M \to \mathbb{R}^N$ such that f_0 agrees with f on A. Let

$$U_0 = \{y||f(y) - f_0(y)| < \delta(y)\}.$$

For each $x \in M - A$, we can find a neighborhood U_x of x such that for all $y \in U_x$ we have

$$\delta(y) > \frac{1}{2}\delta(x)$$

and

$$|f(y) - f(x)| < \frac{1}{2}\delta(x).$$

It follows that for all $y \in U_x$

$$|f(y) - f(x)| < \delta(y).$$

As $\{U_x\}$ is cover of $M - A$, we can choose a countable subcover $\{U_{x_i}\} = \{U_i\}$. Let $\{\rho_0, \rho_i\}$ be a partition of unity subordinate to $\{U_0, U_i\}$ and define

$$\tilde{f}(y) = \rho_0(y)f_0(y) + \sum_{i\geq 1} \rho_i(y)(x_i).$$

We get

$$\begin{aligned}
|\tilde{f}(y) - f(y)| &= \left|\rho_0(y)(\tilde{f}(y) - f_0(y)) + \sum_{\imath\geq 1} \rho_i(y)(\tilde{f}(y) - f(x_i))\right| \\
&\leq \rho_0(y)|\tilde{f}(y) - f_0(y)| + \sum_{i\geq 1} \rho_i(y)|\tilde{f}(y) - f(x_i)| \\
&< \rho(y)\delta(y) + \sum_{i\geq 1} \rho_i(y)\delta(y) = \delta(y).
\end{aligned}$$

Finally, observe that for $a \in A$, $\rho_i(a) = 0$ for all $i \geq 1$ because $\operatorname{supp}(\rho_i) \subset U_i \subset M - A$, hence $\tilde{f}(a) = f_0(a) = f(a)$. □

Definition 1.14. Let M and N be manifolds with boundary. A smooth homotopy $H : M \times I \to N$ from $f = H(-,0)$ to $g = H(-,1)$ is *isotopy* from f to g, if $H(-,t)$ is embedding for every t. We say f and g are *isotopic*.

It is easy to check that Isotopy is equivalence relation. For instance to prove transitivity, let $\rho : [0,1] \to [0,1]$ be a non-decreasing C^∞ function such that

$$\rho(t) = \begin{cases} 0, & \text{if } t \leq 1/3 \\ 1, & \text{if } t \geq 2/3 \end{cases}$$

Now suppose H is isotopy from f to g and K is isotopy from g to h, then J defined by

$$\rho(t) = \begin{cases} H(x, \rho(2t), & \text{if } t \leq 1/2 \\ K(x, \rho(2t-1)), & \text{if } t \geq 1/2 \end{cases}$$

is isotopy from f to g. We will prove the a sharpened version of the theorem for compact manifolds.

Proposition 1.15. *Suppose $i_0, i_1 : M \to \mathbb{R}^N$ are two embeddings, then the stabilizations $i_0 \times 0$ and $i_1 \times 0$.*

Proof. We note the following

(1) $i_0 \times 0$ is isotopic to $i_0 \times i_1$ by isotopy $H_1(x,t) = (i_0(x), ti_1(x))$
(2) $i_0 \times i_1$ is isotopic to $0 \times i_1$ by isotopy $H_2(x,t) = ((1-t)i_0(x), (1-t)i_0(x) + ti_0(x))$
(3) $0 \times i_1$ is isotopic to $i_1 \times 0$ by isotopy $H_3(x,t) = (ti_1(x), (1-t)t_1(x))$

It follows that $i_0 \times 0$ is isotopic to $i_1 \times 0$ as isotopy is transitive. □

1.3. **Tubular Neighborhoods.** The second important construction we need is the following

Definition 1.16. Let $\iota : S \hookrightarrow M$ is an embedding. A *Tubular neighborhood* of ι is a triple (ξ, U, f) of a vector bundle $\xi : E \to S$ and a diffeomorphism $f : E(\xi) \twoheadrightarrow U \subset M$, where U is open neighborhood of S and $0-$section is mapped by f onto N (i.e the following diagram commutes

$$\begin{tikzcd} S \arrow[d, "0_\xi"'] \arrow[dr, hook, "\iota"] & \\ E \arrow[r, "f"] & U \end{tikzcd}$$

Theorem 1.17. *If $S \subset X$ is a closed neat submanifold, then there is a tubular neighborhood of S in X.*

Proof Sketch. We can assume $\iota : S \subset M$ and we identify the zero section with S. We show that there there is an open embedding from a neighborhood U of the zero section of the total space of the normal bundle of S which restricts to identity on the the zero section S. That implies the satement because we can find diffeomorphism $E \to U$ which fixes zero section by rescaling on fibers.The proof is divided into 4 steps.

Step 1. Choose a metric g on M with the following property: there is a collar (W, h) on M, which restricts to a collar on S, such that g restricted to W is $h^*g = \pi_1^*g' + \pi_2^*dt^2$, where g' is a metric on ∂M. The main point is

> (*) Let $\gamma_1 : (a, b) \to \partial M$, and $\gamma_2 : (a, b) \to [0, 1)$ be curves. Then the curve $\gamma = h \circ (\gamma_1, \gamma_2) : (a, b) \to M$ is geodesic if and only if γ_1 and γ_2 are geodesics in $(\partial M, g')$ and $([0, 1), dt^2)$ respectively.

Step 2. Constructing exponential map: Let $\mathcal{N} = \mathcal{N}_S^M = (\pi, N, S)$ be the normal bundle of S in M. For $x \in M$ and $v \in T_xM$, we let γ_v be the unique maximal geodesic γ, with $\gamma(0) = x$ and $\gamma'(0) = v$. The following claim follows from ODE existence theorems and the choice of the metric above:

Claim. There is $\epsilon : S \to (0, \infty)$ and such that the neighborhood $N(\epsilon) = \{(x, v) \in \mathcal{N} | g_x(v, v)^{1/2} < \epsilon(x)\}$ of the zero section of $\mathcal{N}$ satisfies the following : given any $v \in N(\epsilon')$, the geodesic γ_v is defined on $(-2, 2)$. Moreover, the map $\Gamma : N(\epsilon) \times (-2, 2) \to$ defined by $\Gamma(v, t) = \gamma_v(t)$ is smooth.

We define the exponential map

$$\exp : N(\epsilon) \to M \ ; \exp(v) = \Gamma(v, 1).$$

Clearly exp is smooth.

Step 3. Proving that $d(\exp)$ is of maximal rank on S: It suffices to show that the image of $d_x\exp$ is T_xM for each $x \in S$. Fix $x \in S$ and S.

- Take any $v \in \mathcal{N}_x$, then it is clear that $exp(tv) = \gamma_v(t)$ for $t \in \frac{\epsilon(x)}{||v||_g}$. Hence
$$d_x\exp(v) = \frac{d}{dt}(\exp(tv))|_{t=0}) = \frac{d}{dt}\gamma_v(t)|_{t=0} = v,$$
that's v lies in im$d_x$exp as well as its all multiples. In otherwords, $\mathcal{N}_x \subset$ im $d_x$exp
- Observe that $\exp \circ i_0 = i_0$. It follows, that $T_xS \subset$ im $d_xi_0 \subset$ im $d_x$exp.

Now the result follows because of the canonical orthogonal decomposition $T_xM = \mathcal{N}_x \oplus T_x$

Step 4. Constructing the tubular neigborhood: We can see that $\partial N(\epsilon)$ is subset of the total space of $E(\mathcal{N}|_{\partial S})$, and for any (x, v) in this total space γ_v maps into the boundary ∂M because of the choice of the metric above. Hence, by step 3 and Lemma 1.2 that exp is local diffeomorphism which restricts to identity on zero section S. As in collar neighborhood theorem and a scaling argument (to get isomorphism of $N(\epsilon)$ and N) we can conclude. □

Remark 1.18. We would like to say few things which can be easily seen from the proof and otherwise

- Tubular neighborhoods can be made as small as we like. More precisely, working in the same situation of the theorem, if we are given any neighborhood V of S, then there is a tubular neihborhood (f, U, ξ) such that $U \subset V$.
- Let $\iota : S \to M$ be embedding where S and M are boundaryless. Suppose that M is endowed with a Riemannian metric g. Let N denotes the total space of the normal bundle of S in M with respect to g viewed as subbundle of $TM|_{\iota(S)}$. Then there is a neighborhood W such that for certain continuous function $\delta : S \to \mathbb{R}_{>0}$, the exponential map
$$exp : N(\delta) = \{(\iota(x), v) \in N | g_x(v, v)^{1/2} < \delta(x)\} \to W$$
is diffeomorphism. Using this diffeomorphism, we define r to be the $\exp^{-1}$ composed with projection of N. We usually say that (W, r) is *partial normal tubular neighborhood of* ι.

Next we define when two tubular neighborhoods are equivalent

Definition 1.19. Let M be a boundary-less manifold. We say two tubular neighborhoods (f_0, U_0, ξ_0) and (f_1, U_1, ξ_1) are *equivalent* if there is an *isotopy* $F : E_0 \times I \to M$, such that $F_1(E(\xi_1)) = F_0(E(\xi_0))$ and $f_1^{-1}F_1 : E_0 \to E_1$ induces a vector bundle isomorphism $\xi_0 \to \xi_1$.

Theorem 1.20. *Any two tubular neighborhoods of a submanifold without boundary are equivalent.*

Proof. Let $S \subset M$ be a submanifold, with tubular neighborhoods (f_0, U_0, ξ_0) and (f_1, U_1, ξ_1) and write $E_i = E(\xi_i)$.
Step 1. By shrinking the tubular neighborhood (f_0, ξ_0), we prove that (U_0, f_0, ξ) is equivalent to a tubular neighborhood (U_0', f_0', ξ_0') such that $f_0'(E_0') = U_0' \subset f_1(E_1)$, hence we can assume $f_0(E_0) \subset E_1$

To do this we let $V \subset E_0$ be an open neighborhood of the zero section such that $f_0(V) \subset f_1(E_1)$. By possibly shrinking V, we can assume $V = \{z | |z| < \delta(p(z))\}$, where $\delta : M \to \mathbb{R}_+$ is smooth. We take $U_0' = f_0(V)$, $\xi_0' = \xi_0$ and $f_0 : E_0' \to$ defined
$$f_0' : E_0' = E_0 \to U_0';\ f_0'(z) = f_0(\frac{\delta(p(z))}{1 + |z|^2} z)$$
it is then obvious that (f_0', ξ_0') is tubular neighborhood of S satisfying $f_0'(E_0') \subset f_1(E_1)$, and that (f_0, ξ_0) is equivalent to (f_0', ξ_0') by equivalence
$$F : E_0 \times I \to;\ F(z, t) = f_0((1 - t)z + t\frac{\delta(z)}{1 + |z|^2} z)$$
Step 2. We prove (U_0', f_0', ξ_o') is equivalent to (f_1, U_1, ξ_1). Define $g = f_1^{-1} \circ f_0 : E_0 \to E_1$, and note that its derivative restricted S is a bundle morphism
$$dg|_S : TS \oplus \xi_0 \to TS \oplus \xi_1,$$
where we use the canonical identification $TE_i|_S$ with $TS \oplus \xi_i$. Let $(\Phi, \bar{\Phi}) : \xi_0 \to \xi_1$ be the fiber derivative of g along M, so that fiberwise $dg|_S$ takes the form
$$\begin{bmatrix} \mathrm{id} & * \\ 0 & \bar{\Phi} \end{bmatrix}$$
Because $dg|S$ is isomorphism, it follows that $(\Phi, \bar{\Phi})$ is bundle map. Now we define $\tilde{H} : E_0 \times I \to E_1$ to be the straightening homotopy
$$\tilde{H}(v, t) = \begin{cases} \frac{1}{t} g(tv) & \text{if } t \neq 0 \\ \Phi(v) & \text{if } t = 0 \end{cases}$$

We show this is continuous (in fact smooth). It suffices,by considering charts and trivializations, to prove this for $S = \mathbb{R}^k$, and ξ_i is trivial $(n-k)-$bundle on S that's $E_i = \mathbb{R}^n \times \{0\} \times \mathbb{R}^m$. Now the map g takes the following form

$$g : \mathbb{R}^k \times \{0\} \times \mathbb{R}^m \to \mathbb{R}^k \times \{0\} \times \mathbb{R}^m$$

$$g(x, y) = (g_1(x, y), g_2(x, 0, y)) \qquad , g(x, 0) = (x, 0)$$

and $\tilde{H}$ will take the form

$$\tilde{H}(x, y, t) = \begin{cases} (g_1(x, ty), \frac{1}{t} g_2(x, ty)) & \text{if } t \neq 0 \\ \bar{\Phi}(x, y) & \text{if } t = 0 \end{cases}$$

By Hadamard Lemma, there is smooth h_2 such that

$$g_2(x, ty) = g_2(x, 0) + th_2(x, ty) = th_2(x, ty)$$

where $h_2(x, 0) = \partial_2 g_2(x, 0)$. So, $\tilde{H}$ can be given by the formula

$$\tilde{H}(x, y, t) = (g_1(x, ty), h_2(x, ty))$$

because $H(x, y, 0) = \bar{\Phi}(x, y) = (x, \partial_2 g_2(x, 0)) = (g_1(x, 0), h_2(x, 0))$. Define

$$H : E_0 \times I \to E_1; H(x, y, t) = f_1 \tilde{H}(x, y, 1 - t)$$

It is clear H provides equivalence of (f_0, U_0, ξ_0) and (f_1, U_1, ξ_1). Now the theorem follows by transitivity of the equivalence. □

It is conventional to define disc tubular neighborhoods to make notation easier. Given a bundle $\xi = (M, \pi, E)$ with metric g, we can define the assciated disc bundle $D_\epsilon(\xi)$ to be the disc bundle on (M, π, D) with

$$D = \{v \in E | g(v, v)^{1/2} < \epsilon\}.$$

This is in particular a manifold with boundary.

We can prove the following statement using a straightening homotopy as in previous proof.

Theorem 1.21. *Let M and N be manifolds (possibly with boundary) and let $T \subset N$ be a compact neat submanifold. Let $f : M \to N$ be a smooth map transversal to T, and denote the inverse image submanifold by $S = f^{-1}(T)$. Suppose that $\Xi = (g, U, \xi)$ be tubular neighborhood of $S \hookrightarrow M$ and (h, V, η) be a tubular neighborhood of $T \hookrightarrow N$. Let $(g|_D, U', D(\xi))$ be an associated disc bundle of Ξ. Then there is a homotoy $F : M \times I \to N$ such that*

1. *$F_1 = f$,*
2. *$F_0|_D$ restricts to a bundle map $U \to V$ over f, more precisely: there is a bundle map (Ψ, f) such that $F_0|_D = h \circ \Psi \circ g|_D$,*
3. *$F_t = f$ on $S \cup (M - U)$, and*
4. *$F_t(N - T) = M - S$.*

Maps into contractible space. The following lemma essentially follows from definitions

Lemma 1.22. *Let X and Y be topological spaces such that Y is contractible, let Suppose that $A \subset X$ is a cofibration (i.e satisfies homotopy extension property) and $f : A \to X$ is a map. Then $f : A \to Y$ extends to a map $X \to Y$.*

Proof. Since Y is contractible, then there is a homotopy $H : A \times I \to Y$ from a point a constant map. That's there is y_0 such that for all $a \in A$

$$H(a, 0) = y_0 \quad \text{and} \quad H(a, 1) = f(a).$$

As $A \subset X$ is cofibration, then we can extend H to $\tilde{H} : X \times I \to Y$. In particular $H(-, 1) : X \to Y$ is extension of $f : A \to Y$. □

Proposition 1.23. *Let X and Y be topological spaces such that Y is contractible and X is compact Hausdorff. Suppose that $A \subset X$ is a closed subspace which has mapping cylinder neighborhood. Let $f, g : X \to Y$ be maps agreeing on A. Then there is a homotopy between f and g relative to A.*

Proof. We want to prove that $A' = X \times \{0, 1\} \cup A \times I \subset X \times I$ has a mapping cylinder neighborhood. By definition of mapping cylinder neighborhood, we can find closed neighborhood N of A containing a subspace B with $U = N - B$ open neighborhood of A such that there is a continuous map $g : B \to A$ and a homeomorphism $h : M_g \to N$ with $h|_{A \cup B} = \mathrm{id}$.

We define

$$N' := X \times [0, 1/4] \cup X \times [3/4, 1] \cup N \times I.$$

Clearly, N' is closed neighborhood of A'. Let

$$B' := (X - U) \times \{1/4, 3/4\} \cup B \times [1/4, 3/4].$$

It can be checked that $U' = N' - B'$ is open neighborhood of A'. We define $g' : B' \to A'$ as following

$$g'(x,t) = \begin{cases} (x, 0), & \text{if } \ (x,t) \in (X - U) \times \{1/4\} \\ (h([x, 8t - 2]), 0), & \text{if } \ (x,t) \in B \times [1/4, 3/8] \\ (g(x), 4t - \frac{3}{2}), & \text{if } \ (x,t) \in B \times [3/8, 5/8] \\ (h([x, -8t + 6], 1), & \text{if } \ (x,t) \in B \times [5/8, 3/4] \\ (x, 1), & \text{if } \ (x,t) \in (X - U) \times \{3/4\} \end{cases}$$

By pasting lemma, g' is continuous map. Now define $h' : B' \times I \sqcup A' \to N'$ by $h'|_{A'} = \mathrm{id}$ and

$$h'|_{B' \times I}((x,t), s) = \begin{cases} (x, \frac{(1-s)}{4}), & \text{if } \ (x,t) \in (X - U) \times \{1/4\} \\ (h([x, 8ts - 2s], (1 - s)t), & \text{if } \ (x,t) \in B \times [1/4, 3/8] \\ (h([x, s]), 4t - \frac{3}{2}), & \text{if } \ (x,t) \in B \times [3/8, 5/8] \\ (h([x, -8ts + 6s], st), & \text{if } \ (x,t) \in B \times [5/8, 3/4] \\ (x, \frac{3+s}{4}), & \text{if } \ (x,t) \in (X - U) \times \{3/4\} \end{cases}$$

Clearly, h' is continuous and $h'((x,t), 1) = g'(x,t)$, hence h' descends to $\overline{h}' : M_{g'} \to N'$. To prove that $\overline{h}'$ is homeomorhism, we only have to check that this map is bijective because $M_{g'}$ and N' are compact hausdorff. We leave the details to the reader.

Now we prove the statement of the theorem: Define the map $F : A' \to Y$

$$F(x,t) = \begin{cases} f(x), & \text{if } \ t = 0, \\ g(x), & \text{if } \ t = 1, \\ f(a), & \text{if } \ x = a \in A, \end{cases}$$

This map is continuous because f and g agrees on A. Now by the above $A' \subset X \times I$ is cofibration, hence by previous lemma, F extends to homotopy $\tilde{F} : X \times I \to Y$ from f to g. □

1.4. **Intersection theory machinery.**

Definition 1.24. Let $f : M \to N$ be a map, where M, N are manifolds with boundary, with $\partial N = \emptyset$. Suppose that $S \subset N$ is a submanifold, and C is a subset of M. We say f is *transversal* to S *along* C (and write $f \pitchfork_C S$) if

$$d_x f(T_x M) + T_{f(x)} S = T_{f(x)} M$$

for all $x \in C \cap f^{-1}(S)$. If $C = M$, we simply write $f \pitchfork S$, and say f is *transversal* to S.

The importance of the notion of transversality relies on the following theorems:

Theorem 1.25. *Let f, X, Y, S be as in above definition. If*

$$f \pitchfork S \quad \textit{and} \quad \partial f \pitchfork S$$

Then there is a unique manifold structure on $f^{-1}(S)$ turning it a neat submanifold of X such that $\operatorname{codim}_X\ S = \operatorname{codim}_Y\ f^{-1}(S)$ *and* $\partial(f^{-1}(S)) = f^{-1}(\partial S)$.

Proof. Choose any $y \in S$. Since S is a submanifold of Y, then there is a neigborhood V_y of y and submersion of $h_y : V_y \to \mathbb{R}$ such that $h_y^{-1}(0) = V_y \cap S$, and $dh_y(T_y Y) = 0$. Let $U_y = f^{-1}(V_y)$ and $g_y := h_y \circ f : U_y \longrightarrow \mathbb{R}^c$, then g_y is submersion on $g_y^{-1}(0)$, indeed if $g_y(x) = 0$, that's $f(x) = y$ then

$$dh_y(T_x X) = dg_y(d_x f(T_x X)) = dh_y(d_x f(T_x X) + T_y S) = dh_y(T_y Y) = T_0 \mathbb{R}^c$$

By proposition 1.8 $g_y^{-1}(0) = f^{-1}(U_y) \cap f^{-1}(S)$ is a neat submanifold, and since $\{f^{-1}(U_y)\}_{y \in Y}$ covers $f^{-1}(S)$, then by proposition **??** $f^{-1}(S)$ is submanifold of Y. □

Theorem 1.26. *[Thom Transversality Theorem] Let M and N be manifolds, where N is boundaryless. Let $f : M \to N$ be a smooth map. Suppose that S is a submanifold of N and that $C \subset X$ is a closed subset. If $f \pitchfork S$ on C and $\partial f \pitchfork S$ on $C \cap \partial M$. Then f is homotopic to a map g such that $g \pitchfork_C S$, and g agrees with f on a neighborhood of C.*

We need the following lemma

Lemma 1.27. *Let $F : M \times B \to Y$ be smooth map, where Y and B are boundaryless manifolds. Suppose F and ∂F are transversal to S. Then for almost all $b \in B$, the map $f_b = F(-, b)$ and ∂f_b is transversal to S.*

Proof. We have by proposition 1.25, $W = F^{-1}(S)$ is a neat submanifold of M. Let $\pi : W \to B$ denote the restriction of the projection map $M \times B \to B$ to W. By *Sard's theorem*, almost all $b \in B$ are regular values of π and $\partial\pi$.

For any such b, we show that $f_b \pitchfork S$: Let $b \in B$ and $x \in f_b^{-1}(S)$. As $F \pitchfork S$, then

$$dF(T_{(x,b)}(M \times B)) + T_s(S) = T_s(N) \tag{1.28}$$

What we need to show is

$$df_b(T_xM) + T_s(S) = T_s(Y)$$

To do this, let $w \in T_sY$, we want to find $v \in T_x(M)$ such that $df_b(v) - w \in T_s(S)$. From 1.28, we can find $u = (u_1, u_2) \in T_{(x,b)}(M \times B) \equiv T_x(M) \times T_b(B)$, such that

$$d_{(x,b)}F(u) - w \in T_s(S)$$

If we find $u' = (u'_1, u_2)$ such that $d_{(x,b)}F(u') \in T_s(S)$, then setting $v = u_1 - u'_1$ gives

$$\begin{aligned} d_x f_b(v) - w &= d_{(x,b)}F(u_1 - u'_1, 0) - w \\ &= d_{(x,b)}F(u - u') - w \\ &= d_{(x,b)}F(u) - w - d_{(x,b)}F(u') \in T_s(S) \end{aligned}$$

To find u', note that $d_{(x,b)}(\pi) : T_{(x,b)}W = T_xM \times T_bB \to T_bB$ is surjective (by regularity of b), this derivative is the projection $(v_1, v_2) \mapsto v_2$. It follows that there is $u' = (u'_1, u_2) \in T_{(x,b)}(W)$. The fact $dF_{(x,b)}(T_{(x,b)}(W)) = T_s(S)$ completes the proof $\square$

Proof of theorem 1.26. The idea is to perturb f away from C while keeping it fixed near C. To achieve this, we use tubular neighborhoods. Now, by Whitney embedding theorem, we can assume A is submanifold of $\mathbb{R}^N$. Let (g, ξ) be a tubular neighborhood of Y. By a partition of unity argument we can find $\epsilon : Y \to \mathbb{R}_+$, such that ϵ-neighborhood Y^ϵ of A is contained in $g(E(\xi))$. It is easy to find a neigborhood U of C such that $f \pitchfork S$ on U. Let $\lambda : X \to [0, 1]$ be equal to 1 on neighborhood W of C contained in U, and to zero outside U. Write $\rho = \lambda^2$ and let

$$F(x, b) = \pi(f(x) + \epsilon(f(x))b)$$

Define $G(x, b) = F(x, \rho(x)b)$. Let $(x, b) \in G^{-1}(S)$, we have two cases

- If $\rho(x) \neq 0$, then $G(x, -)$ is composition of the diffeomorphism $b \mapsto \rho(x)b$ and $F(x, -)$. But $F(x, -)$ is submersion as it is a composition of two submersions π and the the function $b \mapsto f(x) + \epsilon(f(x))b$. As a consequence $G \pitchfork S$.
- If $\rho(x) = 0$, that's $x \in V$, then by chain rule

  $$d_{(x,b)}G(v, w) = d_{(x,b)}F(v, d_x\rho(v) \otimes b + \rho(x)w)$$

  But $d_x\rho = 2\lambda(x) \cdot d_x\lambda = 0$, hence $d_{(x,b)}G(v, w) = d_xf(v)$. In particular $d_{(x,b)}G$ and d_xf have the same images, so $G \pitchfork S$

By the lemma, there is $b' \in B$, such that $f_b = G(-, b)$ and ∂f_b are transversal to S. Now we let $H : X \times I \to A$ be defined by $H(x, t) = G(x, tb')$. Clearly $H(-, 0) = f$ and $H(-, 1)$ is transversal to S. So H is the intended homotopy and the proof is finished. $\square$

Corollary 1.29. *Let $f : M \to A$ be a smooth map, where A Is boundaryless manifold. Suppose S is submanifold of A, and $\partial f \pitchfork S$. Then f is homotopic to a smooth map $g : M \to A$, such that $\partial f = \partial g$ and $g \pitchfork S$.*

2. Thom-Pontryagin theorem

2.1. **Compactifications and Thom sapce.** We can restate the Thom isomorphism in terms of the what so called *Thom spaces* of bundles. Before defining such a space, we digress to recall the one-point compactification construction and some properties. Recall that given a locally compact Hausdorff (LCH) X, there is a space Y such that

- X is a subspace of Y.
- $Y - X$ is a single point.
- Y is compact Hausdorff.

This space Y is unique up to unique homemorphism fixing X. We choose for each X such Y and denote it by X^*, we usually denote $X^* - X = \{\infty_X\}$ and by abuse of notation we drop the X subscript. The topology on X^* has a simple description: The collection of open sets consists of the open sets U of X and the complements $X^* - K$, where K is compact in X. Note also that X^* comes with canonical base point which is ∞. Now let's recall the functorality properties. The one-point compactification can be made into functor in two ways:

(1) $(-)^*$ is a covariant functor under proper map. Explicitly, let $f : X \to Y$ be proper map of (LCH) spaces, then the functor maps it to $f^* : X^* \to Y^*$ defined by

$$f^*(x) = \begin{cases} f(x), & \text{if } \ x \in X \\ \infty, & \text{if } \ x = \infty. \end{cases}$$

(2) $(-)^*$ is a contravariant functor under open embeddings. Explicitly, let $f : X \to Y$ be an open embedding, then the functor maps it to $f^* : Y^* \to X^*$ defined by

$$f^*(y) = \begin{cases} f^{-1}(y), & \text{if } \ y \in f(X) \\ \infty, & \text{otherwise.} \end{cases}$$

To see that this is continuous, observe that

Note that although we used the same notation for both functors, this will not cause any confusion as it will be clear if the function is proper or open embedding. The two functors are compatible with each other in the sense that if we have a commutative diagram of LCH spaces

$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d, "g"] & Y_1 \arrow[d, "h"] \\ X_2 \arrow[r, "f_2"] & Y_2 \end{tikzcd}$$

with h, g proper and f_1, f_2 open embeddings, then the following diagram is commutative

$$\begin{tikzcd} (X_1)^* \arrow[d, "g^*"] & (Y_1)^* \arrow[l, "(f_1)^*"'] \arrow[d, "h^*"] \\ (X_2)^* & (Y_2)^* \arrow[l, "(f_2)^*"'] \end{tikzcd}$$

Finally, we have the following:

- For compact Hausdorff space X, we have X^* is naturally isomorphic to X_+ (X adjointed to a disjoint point).
- For LCH spaces X, Y, we have isomorphism of based spaces $(X \times Y)^* \cong X^* \wedge Y^*$, where the isomorphism is natural in X and Y. From now on, we we will identify the two spaces.
- For LCH spaces X, Y, we have $(X \sqcup Y)^* \cong X^* \vee Y^*$, where the isomorphism is natural in X and Y. In fact it is given by

$$X \ni x \mapsto x \in X^* \vee Y^*,$$
$$Y \ni y \mapsto y \in X^* \vee Y^*,$$
$$\infty \mapsto \infty.$$

Now, given an n-plane real bundle ξ with total space E, we can form a bundle $Sph(\xi)$ by applying the one point compactification to each fiber. The fibers are spheres with given basepoints, namely the points at ∞. These base points forms a section $i_\infty : B \to Sph(\xi)$. We define

$$\mathrm{Th}(\xi) = Sph(\xi)/i_\infty(B).$$

The result is base point space with base point denoted by ∞.

Remark 2.1. Let ξ, ξ' be vector bundles over B and B' respectively.

(1) We have the following canonical isomorphisms
- $\mathrm{Th}(\epsilon^n) \equiv \Sigma^n B_+$.
- $\mathrm{Th}(\xi \times \xi') \equiv \mathrm{Th}(\xi) \wedge \mathrm{Th}(\xi')$
- $\mathrm{Th}(\xi \oplus \epsilon) \equiv \Sigma\mathrm{Th}(\xi)$

(2) $\mathrm{Th}(-)$ is functorial, where this follows easily by functoraliy of $Sph(-)$. Indeed, a map $(f, \overline{f}) : \xi \to \xi'$ induces $(Sph(f), \overline{f}) : Sph(\xi) \to Sph(\xi')$ where $Sph(f)|_{\pi^{-1}(b)}$ is the one point compactification of $f|_{\pi^{-1}(b)}$, where π denotes the projection of both ξ and $Sph(\xi)$. In fact it is also functorial with respect to (constant rank) bundle morphisms.

(3) If the base space of a bundle ξ is compact, then $\mathrm{Th}(\xi)$ is just the one point compactification of $E(\xi)$.

(4) If the bundle ξ is endowed with a Euclidean metric, we can form the disc bundle $D(\xi)$ and sphere bundle $S(\xi)$. Then ξ is homeomorphic to the quotient $D(\xi)/S(\xi)$

2.2. **Cobordism groups.** Now we define oriented (unoriented) cobordism

Definition 2.2. Let M and N be two smooth compact oriented manifolds, we say M and N to be *oriented cobordant* if there is an oriented manifold X such that ∂X is diffeomorphic as oriented manifold to $M \sqcup (-N)$. If M and N are not oriented, we say M and N are *unoriented cobordant (or just cobordant)* if there is a manifold X such that ∂X is homeomorphic to $M \sqcup N$.

Proposition 2.3. *The relations of (unoriented) cobordism and unoriented cobordism are equivalence relations*

Proof Sketch. (1) (Reflexivity) Let M be a manifold, then $M \sqcup -M$ is diffeomorphic to the boundary of $[0,1] \times M$.

(2) (Symmetry) If M is cobordant to M', then there is X such that $\partial X \cong M \sqcup (-M)$, hence $\partial(-X) \cong (-M) \sqcup M$

(3) (Transitivity) Suppose that M is cobordant to M' and M' is cobordant to M''. Write $M \sqcup (-M') \cong \partial X$ and $M' \sqcup (-M'')$, then we can use the collar theorem to glue the smooth structures of X and Y along M' to obtain a new smooth manifold with boundary $M \sqcup (-M')$.

□

We denote the set of all oriented (unoriented) cobordism classes of n-dimensional Ω_n^{SO} (resp. Ω_n^O). It is easy to see that Ω^{SO} and Ω^O form additive groups, where disjoint union define the addition, so that $[M_1]+[M_2] := [M_1 \sqcup M_2]$. The zero element of this group is given by the vacuous manifold $\emptyset$.

Moreover, the cartesian product of manifolds $M_1^n, M_2^m \mapsto M_1^m \times M_2^n$ gives rise to bilinear, associative product operation

$$\Omega_n \times \Omega_m \to \Omega_{n+m}$$

And thus $\Omega_* = (\Omega_0, \Omega_1, \cdots)$ of oriented (unoriented) cobordism groups has the structure of a graded ring. This ring has two sided identity element $1 = [*]$. Moreover, we know that $M_1^m \times M_2^n$ is isomorphic as oriented (unoriented) manifold to $M_2^n \times M_1^m$. So Ω_* is graded commutative ring.

2.3. **Statement of Thom-Pontrjagin theorem.** We begin by defining Thom spectra. Define the spectrum MO to be the spectrum associated to the ring pre-spectrum TO to be defined as following:

- $TO(q) = \mathrm{Th}(\gamma^q)$
- We have a bundle map $\gamma^n \oplus \epsilon \to \gamma^{q+1}$ which defines the structure maps $\sigma_q : \Sigma TO(q) \equiv \mathrm{Th}(\gamma^n \oplus \epsilon) \to TO(q+1)$, this bundle map lies over the inclusion $i_q : BO(q) \to BO(q+1)$

Similarly, define TSO spectrum by replacing γ with $\tilde{\gamma}$. We recall that homotopy groups of pre-spectrum T_q is given by

$$\pi_n(T) = \lim_{q\to\infty} \pi_{n+q}(T_q)$$

where the limits are taken over maps

$$\pi_{n+q}(T_q) \xrightarrow{\Sigma} \pi_{n+q+1}(\Sigma T_q) \xrightarrow{(\sigma_q)_*} \pi_{n+q+1}(T_{q+1})$$

We often denote the spectra associated to TO by MO. The homotopy groups of spectra are the same of the pre-spectrum they originated from. For completeness we give a breif idea about the ring structure of TO. We choose once and for all bundle maps $\gamma^m \times \gamma^n \to \gamma^{m+n}$ for all pairs (m, n). One can define the maps explicitly by taking

$$((V, v), (W, w)) \mapsto (V \times W, (v, w)),$$

where $V \subset \mathbb{R}^\infty$ containing vector v is m-plane and $W \subset \mathbb{R}^\infty$ is n-plane containing vector w. It is understood that $V \times W \subset \mathbb{R}^\infty \times \mathbb{R}^\infty \cong \mathbb{R}^\infty$, where the last identification can be taken to be

$$(a_0, a_1, \cdots), (b_0, b_1, \cdots) \mapsto (a_0, b_0, a_1, b_1, \cdots).$$

We can see that $\mathrm{Th}(\gamma^m \times \gamma^n)$ is canonically homeomorphic to $\mathrm{Th}(\gamma^m) \wedge \mathrm{Th}(\gamma^n)$, hence $\gamma^m \times \gamma^n \to \gamma^{m+n}$ defines $\sigma_{m,n} : TO(m) \wedge TO(n) \to TO(m+n)$. The collection $\{\sigma_{m,n}\}$ can be checked to form a pre-spectrum structure on TO and hence a ring spectrum structure on MO. Thom proved the following amazing result in his paper [?Thom]

Theorem 2.4 (Thom-Pontryagin Isomorphism)**.** *We have a ring isomorphism*

$$\Omega_n^O \cong \pi_n(MO)$$

2.4. **The proof.** We identify the based circle S^1 with $\mathbb{R}$, and hence the sphere $S^n = S^1 \wedge \cdots \wedge S^1 = \mathbb{R}^* \wedge \cdots \wedge \mathbb{R}^* = (\mathbb{R}^n)^*$. Define the *Pontryagin collapse map* $t : \Omega_* \to \pi_*(TO)$ as following. Starting from cobordism class $[M]$, choose an embedding $\iota : M \to \mathbb{R}^{n+r}$ with r sufficiently large by embedding theorem 1.11, then choose a smooth tubular neighborhood $\Xi = (f, U, \xi)$ of ι by theorem 1.17, and finally pick a smooth classifying map $e : \xi \to \gamma^r(\mathbb{R}^N)$ for large enough N. Regarding f as open embedding $E(\xi) \hookrightarrow \mathbb{R}^{n+r}$, we denote by $c = c(M, \iota, \Xi, e)$ (c for collapse) the composition

$$S^{n+r} = (\mathbb{R}^{n+r})^* \xrightarrow{f^*} \mathrm{Th}(\xi) \longrightarrow \mathrm{Th}(\gamma^r) \quad .$$

Here the first arrow is the stereographic projection, and the final arrow is the Thom map of $\xi \to \gamma^r$ which is composition of classifying map $e : \xi \to \gamma^r(\mathbb{R}^N)$ and the canonical bundle map $\gamma^r(\mathbb{R}^N) \to \gamma^r$, we will denote the latter composition by e too. We remark here that c satisfies the following properties:

(1) The map c factors through a map $S^{n+r} \to \mathrm{Th}(\gamma^r(\mathbb{R}^N))$.
(2) $c : S^{n+r} \to \mathrm{Th}(\gamma^r(\mathbb{R}^N))$ is smooth on $c^{-1}(\mathrm{Th}(\gamma^r(\mathbb{R}^N) - \{\infty\}) = f(E) = U$
(3) $c|_U$ is transversal to the zero section $G_r(\mathbb{R}^N)$, and $c^{-1}(G_r(\mathbb{R}^N)) = M$.

Now, let $t([M])$ be the element of $\pi_n(MO) = \lim_{r\to\infty}\pi_{n+r}(\mathrm{Th}(\gamma^r))$ represented by $c(i, M, f, \xi, e)$. Note that t may not well-defined map $\Omega_n \to \pi_n(TO)$ as it may depend on the choice of M in its cobordism class, the embedding ι, the tubular neighborhood $\Xi = (f, U, \xi)$ and the classifying map e. The proof is divided into 4 steps to show that t indeed gives a well defined ring isomorphism.

Step 1. We prove that t is well defined. That's we prove the statement :

(*) for cobordant $n-$manifolds M_0 and M_1, the map $c(M_0, \iota_0, \Xi_0, e_0)$ represents the same element as $c(M_1, \iota_1, \Xi_1, e_1)$, where $\iota_j : M_j \to \mathbb{R}^{n+r_j}$ are embeddings, $\Xi_j = (f_j, U_j, \xi_j)$ are tubular neighborhoods of ι_j, and $e_j : \xi_j \to \gamma^{r_j}$ are smooth classifying bundle isomorphisms.

We do this in a series of further steps:

(1) (*) is true when $M_0 = M_1 = M$, $\iota_1 = \iota_2 = \iota$, $\Xi_1 = \Xi_2 = \Xi = (f, U, \xi)$: we know that $e_0, e_1 : \xi \to \gamma^r$ are homotopic through bundle maps, that's there is a bundle map $(H, \overline{H}) : \xi\times I \to \gamma^r$. The composition

$$S^{n+r} \wedge I_+ \xrightarrow{\cong} (\mathbb{R}^{n+r} \times I)^* \xrightarrow{f^*} \mathrm{Th}(\xi \times I) \xrightarrow{\mathrm{Th}(H)} \mathrm{Th}(\gamma^r)$$

gives us based homotopy from $c(M, \iota, \Xi, e_0)$ to $c(M, \iota, \Xi, e_1)$

(2) (*) is true when $M_1 = M_2 = M$, $\iota_1 = \iota_2 = \iota$: We denote by E_0 the total space of ξ_0. Let $F : E_0\times I \to \mathbb{R}^{n+r}$ be an equivalence of the tubular neighborhoods **??**, and let $\hat{F} : E_0\times I \to \mathbb{R}^{n+r}\times I$ be its track. We know that $\hat{F}$ is open embedding, hence we can apply $(-)^*$ functor to get a map

$$G = (\hat{F})^* : S^{n+r} \wedge I_+ \cong (\mathbb{R}^{n+r})^* \wedge I_+ \longrightarrow \mathrm{Th}(\xi) \wedge I_+$$

where we have used the natural identification $(X \times I)^* \cong X^* \wedge I_+$ mentioned earlier. By composing this with projection on first factor

$$\pi : \mathrm{Th}(\xi_0) \wedge I_+ = \frac{\mathrm{Th}(\xi_0) \times I}{\{\infty\} \times I} \longrightarrow \mathrm{Th}(\xi_0),$$

we get a based homotopy H. Further composing with $\mathrm{Th}(e_0)$, we get a based homotopy $J : S^{n+r} \wedge I_+ \to \mathrm{Th}(\gamma^r)$. Clearly, $J_0 = c(M, \iota, \Xi_0, e_0)$. And since $F = f_1\Phi$, for some bundle isomorphism $(\Phi, \bar{\Phi}) : \xi_0 \to \xi_1$ then

$$\begin{aligned} J_1 = \mathrm{Th}(e_0) \circ H_1 = \mathrm{Th}(e_0) \circ (F_1)^* &= \mathrm{Th}(e_0) \circ (\Phi)^* \circ (f_1)^* \\ &= \mathrm{Th}(e_0) \circ \mathrm{Th}(\Phi^{-1}) \circ (f_1)^* \\ &= c(M, \iota, \Xi_1, e_0\bar{\Phi}^{-1}) \end{aligned}$$

Hence $c(M, \iota, \Xi_0, e_0) \simeq c(M, \iota, \Xi_1, e_0\bar{\Phi}^{-1})$ which by (1) represents the same element as $c(i, M, \Xi_1, e_1)$ (Note that $e_0\bar{\Phi}^{-1}$ is a classifying map of ξ_1).

(3) (*) is true when $M_0 = M_1 = M$: we start with data for (ι_j, Ξ_j, e_j) for $j = 0, 1$ of embeddings $\iota_j : M \to \mathbb{R}^{n+r_j}$, tubular neighborhoods (f_j, U_j, ξ_j) of ι_j, and classifying map e_j of ξ_j respectively. The goal is to prove that the elements $c(M, \iota_j, \xi_j, e_j)$ represents the same class in the homotopy group.

- We claim that $(\sigma_r)_* \, \Sigma c(M, \iota, \Xi, \xi, e) = c(M, \iota, \Xi', e')$, where $\Xi' = (f', U', \xi')$ is tubular neighborhood of M in $\mathbb{R}^{n+r+1}$, ι' is an embedding $M \hookrightarrow \mathbb{R}^{n+r+1}$ (actually the stablization of ι, and e' is classifying bundle map $e' : \xi' \to \gamma^{r+1}$. Indeed,

  $$\Sigma c(i, M, f, \xi, e) = \Sigma \mathrm{Th}(e') \circ \Sigma f^* = \Sigma \mathrm{Th}(e) \circ (f^* \wedge \mathrm{id}_{S^1}) = \mathrm{Th}(e \times \mathrm{id}) \circ (f \times \mathrm{id}_{\mathbb{R}})^*$$

  it follows that we can take $\iota'(x) = (\iota(x), 0)$, $\Xi' = (f \times \mathrm{id}_{\mathbb{R}}, U \times \mathbb{R}, \xi \oplus \epsilon)$, and e' the composition of $e \oplus \mathrm{id}$ with the bundle map $\gamma^r \oplus \epsilon \to \gamma^{r+1}$. We conclude that we can assume r_j is as large as we want.
- Take $r_0 = r_1$ sufficiently large and $H : M \to \mathbb{R}^{n+r}$ be isotopy from ι_1 to ι_2, which exists by proposition 1.15. Let $\hat{H} : M \times I \to \mathbb{R}^{n+r} \times I$ denote track of such isotopy. We see that $\hat{H}$ is a neat embedding. Hence, we can find a "neat" tubular neighborhood $\hat{\Xi} = (\hat{f}, \hat{U}, \hat{\xi})$ of $\hat{H}$. We denote by $\hat{\Xi}_0$ and $\hat{\Xi}_1$ the induced tubular neighborhoods of ι_1 and ι_2 respectively (obtained by restricting $\hat{\Xi}$ to $M \times 0$ and $M \times 1$, In other words, we have the commutative diagram

  $$\begin{tikzcd}
  E(\hat{\xi_0}) \arrow[r, "\hat{f}_0"] \arrow[d, "\text{bundle map}"'] & \mathbb{R}^{n+r} \arrow[d, "(\mathrm{id},0)"] \\
  E(\hat{\xi}) \arrow[r, "\hat{f}"] & \mathbb{R}^{n+r} \times I \\
  E(\hat{\xi_1}) \arrow[r, "\hat{f}_1"] \arrow[u, "\text{bundle map}"] & \mathbb{R}^{n+r} \arrow[u, "(\mathrm{id},1)"']
  \end{tikzcd} .$$

  Let $\hat{e} : \hat{\xi} \to \gamma^r(\mathbb{R}^N)$ be a bundle map, then the composition of $\hat{\xi_j} \to \hat{\xi}$ and $\hat{e}$ is a bundle map $\hat{e_j}$. We get a diagram

  $$\begin{tikzcd}
  (\mathbb{R}^{n+r})^* \arrow[r, "(\hat{f}_0)^*"] \arrow[d, "\mathrm{id}\wedge 0"'] & \mathrm{Th}(\hat{\xi_0}) \arrow[d] \arrow[dr, "\mathrm{Th}(\hat{e_1})"] & \\
  (\mathbb{R}^{n+r})^* \wedge I_+ \arrow[r, "(\hat{f})^*"] & \mathrm{Th}(\hat{\xi}) \arrow[r, "\mathrm{Th}(\hat{e})"] & \mathrm{Th}(\gamma^r) \\
  (\mathbb{R}^{n+r})^* \arrow[r, "(\hat{f}_1)^*"] \arrow[u, "\mathrm{id}\wedge 1"] & \mathrm{Th}(\hat{\xi_1}) \arrow[u] \arrow[ur, "\mathrm{Th}(\hat{e_1})"'] &
  \end{tikzcd}$$

  This means that there is a homotopy from $c(M, \iota_0, \hat{\Xi}_0, \hat{e_0})$ and $c(M, \iota_1, \hat{\Xi}_1, \hat{e_1}$, we conclude by (2) above.

(4) Finally (*) is true: In view of the revious parts, what we need to show is that for cobordant M_0 and M_1 of dimension n, we have $c(M_0, \iota_0, f_0, \xi_0, e_0)$ represents the same class as $c(M_1, \iota_1, f_1, \xi_1, e_1)$, where $\iota_j : M_j \to R^{n+r_j}$ is embedding, Ξ_j is tubular neighborhoodof ι_j and e_j is classifying map of ξ_j are all to be chosen. Let W be an $n+1$ manifold, such that $\partial W = M_0 \sqcup M_1$. Using proposition 1.12 obtain a neat embedding $\iota : W \to \mathbb{R}^{n+r} \times [0, 1]$ such that $\iota(M_k) \subset \mathbb{R}^{n+r} \times \{k\}$ that.

Let $\hat{\Xi}$ be a tubular neighborhood of ι, and let $\hat{\Xi}_j$ be the induced tubular neighborhood of its restriction to ι_j (defined by $\iota(x) = (\iota_j(x), j)$ for $x \in M_j$). We then proceed as in (3) above to conclude.

Step 2. We prove that t is ring homeomorphism.

- Let M_0 and M_1 be manifolds of same dimension n and $M = M_0 \sqcup M_1$, we show that

$$t([M]) = t([M_0]) + t([M_1]). \tag{2.5}$$

We take $c_j := c_j(M_j, \iota_j, f_j, e_j)$ where

$$\iota_0(M_0) \subset \mathbb{R}^{n+r-1} \times \mathbb{R}_{>0}$$

and $\Xi_0 = (f_0, U_0, \xi_0)$ chosen such that $U_0 \subset \mathbb{R}^{n+r-1} \times \mathbb{R}_{>0}$. We define ι_1 and Ξ_1 similarly with $\mathbb{R}_{>0}$ replaced by $\mathbb{R}_{<0}$. Now we let ι be the open embedding

$$M \xhookrightarrow{\iota_0 \sqcup \iota_1} \mathbb{R}^{n+r-1} \times \mathbb{R}_{>0} \sqcup \mathbb{R}^{n+r-1} \times \mathbb{R}_{<0} \hookrightarrow \mathbb{R}^{n+r} \; .$$

Next define tubular neighborhood $\Xi = (f, U, \xi)$ of ι by taking ξ to be the bundle over M with total space $E(\xi_0) \sqcup E(\xi_1)$, $U = U_0 \cup U_1$ and $f = f_0 \cup f_1$. We get the following commutative diagram

$$\begin{tikzcd}
(\mathbb{R}^{n+r})^* \arrow[r] \arrow[d] & ((\mathbb{R}^{n+r-1}\mathbb{R}_{>0}) \sqcup (\mathbb{R}^{n+r-1} \times \mathbb{R}_{<0}))^* \arrow[d] \\
E(\xi)^* \arrow[r] & (E(\xi_0) \sqcup E(\xi_1))^*
\end{tikzcd}$$

We have an isomorphism $\mathbb{R}^{n+r-1} \times \mathbb{R}_{>0} \cong \mathbb{R}^{n+r}$ defined by $(x,t) \mapsto (x, \log(t))$, and similarly $\mathbb{R}^{n+r-1} \times \mathbb{R}_{>0} \cong \mathbb{R}^{n+r}$ defined by $(x,-t) \mapsto (x, \log(t))$. Denote the f_0' to be the composition of f_0 with the first of the above isomorphism, similarly define f_1'. We use the fact that $(X \sqcup Y)^* = X^* \wedge Y^*$ and get the commutative diagram (the upper square is commutative only up to homotopy)

$$\begin{tikzcd}
S^{n+r} \arrow[rr, "\text{collapse along equator}"] \arrow[d, "\cong"] & & S^{n+r} \vee S^{n+r} \arrow[d, "\cong"] \\
(\mathbb{R}^{n+r})^* \arrow[r] \arrow[d] & (\mathbb{R}^{n+r-1} \times \mathbb{R}_{>0})^* \vee (\mathbb{R}^{n+r-1} \times \mathbb{R}_{<0})^* \arrow[r, "\cong"] \arrow[d, "(f_0)^* \vee (f_1)^*"] & (\mathbb{R}^{n+r})^* \vee (\mathbb{R}^{n+r})^* \arrow[dl, "(f_0')^* \vee (f_1')^*"] \\
\mathrm{Th}(\xi) \arrow[r] & \mathrm{Th}(\xi_0) \vee \mathrm{Th}(\xi_1) \arrow[d, "\mathrm{Th}(e_0) \vee \mathrm{Th}(e_1)"] & \\
& \mathrm{Th}(\gamma^r) &
\end{tikzcd}$$

This diagram implies the result by definition of addition in homotopy group. That's by letting $e = e_0 \cup e_1$, we have by tracing the diagram

$$c(M, \iota, \Xi, e) \simeq c(M_0, \iota_0, \Xi_0, e_0) + c(M_1, \iota_1, \Xi_1, e_1).$$

- Start with n_0-dimensional manifold M_0 and m_0-dimensional manifold M_1. We want to prove that

$$t([M_0 \times M_1]) = t([M_0]) \cdot t([M_1]).$$

We let $\iota_j : M_j \hookrightarrow \mathbb{R}^{n_j+r_j}$ be embedding $\Xi_i = (f_j, U_j, \xi_j)$ be tubular neighborhoods of ι_j, and let e_j be classifying map of ξ_j. Obviously, $\iota_0 \times \iota_1$ is embedding of $M_0 \times M_2$ into $\mathbb{R}^{n_0+r_0} \times \mathbb{R}^{n_1+r_1} = \mathbb{R}^{n_0+r_0+n_1+r_1}$, and $(f, U, \xi) = (f_1 \times f_2, \xi_0 \times \xi_1)$ is tubular neighborhood of i. We let $c := c(i, M_1 \times M_2, f, \xi, \sigma_{r_0,r_1} \circ e)$. The following diagram is commutative (the leftmost square is commutative up to homotopy only)

$$\begin{tikzcd}
S^{n_0+n_1+r_0+r_1} \arrow[r, "\cong"] \arrow[d, "\cong"] & (\mathbb{R}^{n_0+n_1+r_0+r_1})^* \arrow[r, "f^*"] \arrow[d, "\cong"] & \mathrm{Th}(\xi) \arrow[r, "\mathrm{Th}(e)"] \arrow[d, "\cong"] & TO(n+m) \\
S^{n_0+r_0} \wedge S^{n_1+r_1} \arrow[r, "\cong"] & (\mathbb{R}^{n_0+r_0})^* \wedge (\mathbb{R}^{n_1+r_1})^* \arrow[r, "f_1^* \wedge f_2^*"] & \mathrm{Th}(\xi_1) \wedge \mathrm{Th}(\xi_2) \arrow[r, "\mathrm{Th}(e_1)\wedge\mathrm{Th}(e_2)"] & TO(r_0) \wedge TO(r_1) \arrow[u, "\mu_{r_0,r_1}"']
\end{tikzcd}$$

The map $S^{n_0+n_1+r_0+r_1} \to TO(r_0 + r_1)$ obtained by composing maps of top row is c, which represents $t([M_1] \cdot [M_2])$, while that obtained by composing the left vertical map, bottom row and right vertical map is $c_1 \cdot c_2$, which represents $t([M_1]) \cdot t([M_2])$.

Step 3. We prove t is injective. Suppose that $t([M]) = 0$, then using part (3) in the first step we can find ι,Ξ and e such that $c = c(M, \iota, \Xi, e)$ is based homotopic to the constant map $S^{n+r} \to \{\infty\} \subseteq \mathrm{Th}(\gamma^r)$, hence there is an extension $H : D^{n+r+1} \to \mathrm{Th}(\gamma^r)$ of c to D^{n+r+1}. Since D^{n+r+1} is compact, then the image of H lies in $\mathrm{Th}(\gamma^r(\mathbb{R}^N)) =: T$ for some N. Write $E = E(\gamma^r(\mathbb{R}^N)) = T - \{\infty\}$ and $V = H^{-1}(T - \{\infty\})$.

<u>Claim.</u> H is homotopic to a map G such that

(1) $H|_{S^{n+r}} = G|_{S^{n+r}}$,
(2) $H^{-1}(\infty) = G^{-1}(\infty)$,
(3) and $G|_V$ is smooth map $V \to E(\gamma^r(\mathbb{R}^N))$.

We know that there is proper embedding $j : E \to \mathbb{R}^{N'}$ for large N', and that this implies $j \circ H|_V$ is proper. Let (W, π) be a partial normal tubular neighborhood **??** of j, in particular for any $x \in W$. $(x - j(\pi(x))) \perp T_{j(\pi(x))}E$. It is easy to find continuous $\epsilon : E \to \mathbb{R}^n$ such that $\epsilon < 1$

$$E^\epsilon := \{y \in \mathbb{R}^{N'} | |y - j(z)| < \epsilon(z) \text{ for some } \ z \in E\} \subset W.$$

By Whitney approximation theorem 1.13 and the fact that U is closed subset of V such that $(H|_V)|_U = c|_U$ is smooth (by property (1) of c above), there is $H' : V \to \mathbb{R}^{N'}$ such that

- $|j \circ H|_V - H'| < \epsilon \circ H|_V$, and
- $H'|_U = H|_U$.
- H' is smooth

Define $G' : V \to E$ by $G'(x) = \pi(H'(x))$, this is well defined by choce of ϵ. It follows that $j(G'(x))$ is the closest point on $j(E)$ to $H'(x)$ (otherwise, the closest point $j(z) \neq j(\pi(x))$ will satisfy

$$(j(z) - j(G'(x))) \perp T_{j(z)}E$$

which contradicts the fact that W is normal tubular neighborhood).Now G' satisfies

- G' is smooth.
- G' is proper. Indeed, by triangle inequality, the fact about $j \circ G'$ mentioned above, $\epsilon < 1$ and the first property of H', we get $|j \circ G' - j \circ H|_V| < 2$. The thesis follows by the fact that $H|_V$ is proper.

- $G'|_U = H|_U$. Indeed, for x in U, we have
$$j^{-1}(\pi(H'(x))) = j^{-1}(H'(x)) = j^{-1}(j(H'(x)) = H(x).$$

Finally, the intended G will be
$$G(x) = \begin{cases} G'(x) & \text{for } x \in V \\ \infty & \text{otherwise} \end{cases}$$
which clearly satisfies all the properties and is continuous exactly because G' is proper. Now it remains to define the homotopy between H and G'. For that take $\mathbb{G} : D^{n+r+1} \times I \to T$ defined by
$$\mathbb{G}(x,t) = \begin{cases} \pi(tj(H|_V(x)) + (1-t)H'(x)) & \text{for } x \in V \\ \infty & \text{otherwise} \end{cases}.$$
By techniques similar to above, this is indeed continuous.

Next, we will apply Thom transersality theorem to obtain a submanifold W of D^{n+r+1} such that $\partial W = M$ and hence finishing the step. For this let W be a precompact open neighborhood of zero section of zero section $G_r(\mathbb{R}^N)$ of E. Let
$$C := (G|_V)^{-1}(E - W) \cup U \subset V.$$
Clearly, C is closed subset of V. Moreover (in the notation of appendix on differential topology),
$$(G|_V) \pitchfork_C G_r(\mathbb{R}^N) \quad , \quad c|_U = \partial(G|_V) \pitchfork_C G_r(\mathbb{R}^N),$$
where the first transversality is trivial and the second one follows from property (3) of c. Now the transversality theorem implies that there is $K : V \to E$ which is homotopic to $G|_V$ such that

- $K|_C = G|_C$ (in particular $\partial K = c$)
- and $K \pitchfork G_r(\mathbb{R}^N)$.

Define $J : D^{n+r+1} \to T$ by
$$J(x) = \begin{cases} K(x), & \text{for } x \in V \\ \infty, & \text{otherwise} \end{cases}$$
From the first property of K and definition of C, it follows that J is continuous. Now, we have $J^{-1}(G_r(\mathbb{R}^N)$ is compact subset of V and on the other hand it is equal $K^{-1}(G_r(\mathbb{R}^N)$ which is submanifold of V of boundary
$$(\partial K)^{-1}(G_r(\mathbb{R}^N)) = c^{-1}(G_r(\mathbb{R}^N)) = M.$$

Step 4. We prove that t is surjective. Let $f : S^{n+r} \to \mathrm{Th}(\gamma^r)$ be a based map representing an element of $\pi_r(MO)$. Because S^{n+r} is compact, then the image of f lies in $\mathrm{Th}(\gamma^r(\mathbb{R}^N))$ for large enough N. Let $V = f^{-1}(E(\gamma^r(\mathbb{R}^N)))$. By arguments similar to the above step, we can assume by homotoping f, that $f|_V$ is smooth and $f|_V$ is transversal to $G_r(\mathbb{R}^N)$.

Write $M = f^{-1}(G_r(\mathbb{R}^N))$ which is compact submanifold of S^{n+r} and pick a tubular neighborhood $\Xi = (g, U, \xi)$ such that $U \subset \overline{U} \subset V$. The goal is to prove that f is based homotopic to $c = c(M, incl., \Xi, e)$, where e is to be determined. Let $(g|_D, U', D(\xi))$ be an associated disc tubular neighborhood of Ξ. The situation is as following: We have

- a map $f|_V : V \to E(\gamma^r(\mathbb{R}^N))$,

- a submanifold (the zero section) $G_r = G_r(\mathbb{R}^N)$ of the target such that $f|_V \pitchfork G_r$ with $(f|_V)^{-1}(G) = M$,
- and tubular neighborhood Ξ (resp. $(id, E(\gamma^r(\mathbb{R}^N)), \gamma^r(\mathbb{R}^N))$ of M in V (resp. G in $E(\gamma^r(\mathbb{R}^{\mathbb{N}}))$.

This is the situation of theorem 1.21, hence we can find a bundle map $e = (\Psi, \overline{\Psi}) : \xi \to \gamma^r(\mathbb{R}^N)$ such that there is homotopy $H : V \times I \to E(\gamma^r(\mathbb{R}^N))$

(1) $H_0|_D = \Psi \circ (g^{-1})|_D$,
(2) $H_1 = f|_V$,
(3) $H_t = f|_V$ on $M \cup V - U$,
(4) and $H_t(E(\gamma^r(\mathbb{R}^N)) - G_r) = H_t(V - M)$

Using property (3), we find that H extends to based homotopy $K : S^{n+r} \times I \to \mathrm{Th}(\gamma^r(\mathbb{R}^N))$ from f by letting $K(x,t) = f(x)$ for x outside V. We write $f' = H_0$, and note that for $x \in D$

$$f'(x) = \Psi(g^{-1}(x)) = \mathrm{Th}(e)(g^{-1}(x)) = \mathrm{Th}(e)(g^*(x)) = c.$$

That's

$$f'|_D = c|_D \tag{2.6}$$

Now we apply a trick, observe that for any bundle η over space compact space B, the thom space $\mathrm{Th}(\eta) - B$ (where B is viewed as zero section) is contractible. Indeed, a deformation retraction to ∞ can be given by homotopy

$$G(x,t) = \begin{cases} \frac{1}{1-t}x, & \text{if} x \neq \infty \text{ and } 0 \leq t < 1 \\ \infty, & \text{if } t = 1 \text{ or } x = \infty. \end{cases}$$

By property (4) of homotopy H, we know $f'|_{S^{n+r}-\mathrm{int}(D)}$ maps into $\mathrm{Th}(\gamma^r(\mathbb{R}^N)) - G_r$, and similarly for c. By equation 2.6, we conclude that $f' = c$ on $\partial D \cup \{\infty\}$. Applying theorem 1.23 to $f'|_{S^{n+r}-\mathrm{int}(D)}$ and $c|_{S^{n+r}-\mathrm{int}(D)}$ considered as maps to contractible space $\mathrm{Th}(\gamma^r(\mathbb{R}^N)) - G_r$ with subset Z of $S^{n+r} - \mathrm{int}(D)$ being $\partial D \cup \infty$, we get

$$f'|_{S^{n+r}-\mathrm{int}(D)} \simeq c|_{S^{n+r}-\mathrm{int}(D)} \quad \mathrm{rel}(\partial D \cup \{\infty\}).$$

Gluing this to the equation 2.6, we obtaind based homotopy from f' to c.